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From point through density valuation to individual risk assessment in the discounted cash flows method

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Abstract
We review the developments and practice of the discounted cash-flow method in finance with an intermediate goal of presenting parsimonious methods of generating density valuation rather than point forecasts. Our ultimate aim is to select, propose and discuss some density-based risk measures that may be used by appraisers and investment analysts when conducting DCF valuation for broad group of heterogeneous (by risk appetite) final users or investors. Such a toolbox may be applied directly by the latter group without necessity to rely on aggregated point valuations and recommendations.

Keywords:
discounted cash-flows, density forecasts, downside risk measures, Monte Carlo simulation

JEL Classification
G17, G12, G11

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1 Introduction

I'd rather be vaguely right, than precisely wrong

John Maynard Keynes

The discounted cash flow method (DCF) in firms' or financial projects' valuation is well-established amongst many investment firms, analysts and final users despite equally broad group of critics. The popularity of this toolbox is propelled by its simplicity and its agreement with a natural, well grounded in financial world, concept of discounting. On the other hand, the usual product of a DCF valuation being a single number is prone to subjective assumptions made during analysis, leaving the final user with the sensitivity tables at best and little clue about risk from such a point forecast. Yet another important argument against the DCF usage is an overly share of so called *terminal value* or *residual value* in the final valuation point forecast. Paradoxically, a part of valuation that supposedly is *residual*, in fact in many cases contributes to more than a half of the forecasted value.

The group of DCF's final users has grown and it includes now not only professional investors but also retail clients, who obviously are less prepared to analyse mathematics and economics behind each recommendation based on classic DCF. In spite of undoubted regulatory effort to increasing transparency and shedding *missell* practises during last decade or two, there is still a room for improvement in the field of communication between professional, hence better informed and quipped, investment managers or analysts and their clients or any stakeholder from public domain. We believe that especially point valuations as a result of classical deterministic DCF method may be misinterpreted by retail investors, giving them undue confidence in certain price direction without proper specification of risks involved. Increased retail investors' protection through better information and risk assessment of investments is the main motivation for this research, nevertheless we posit that still some professional users of DCF method may benefit from the solutions proposals listed here.

During the period that elapsed from the beginning of DCF's wide popularity (mid of 60's, after Modigliani and Miller (1958)) we observed the outburst of applied stochastic methods in mathematical finance, with special focus on financial derivatives' modelling and valuation. Impressive number of papers have been authored on practical usage of different stochastic differential equations (SDE) in pricing models of contingent claims, started with seminal works of Black and Scholes (1973) and Merton (1973). What made it possible was the notion of *risk-neutral probability measure* under which the expectations of future price realisations are taken. The assumptions of complete and arbitrage free markets led to theorems of such a probabilistic measure's existence making the derivatives pricing calculations

henceforth relatively simple.
Coincidental to the derivatives' model developments, the increase of computational power of average PC influenced also other scientific branches as statistics, probability theory, econometrics or numerical analysis and contributed dramatically to their expansion. Having this in mind it is odd that we still are attached to point valuation, not the full density forecast in DCF. The core concept and practical side of DCF method primarily stayed in deterministic domain probably because the method is positioned in so called *physical risk measure*, it is usually impossible to implement no arbitrage assumption, it was meant to produce one number - a point valuation to be used for accounting or investment purposes and the dominant group amongst final users of the method were financial services professionals.
In one of the definitions of *risk* after Carmichael and Balatbat (2008) we read: "*risk [is] an exposure to the chance of occurrences of events adversely or favourably affecting the investment as a consequence of uncertainty*". It is hard to imagine that point forecasts or valuations are enough nowadays, especially when the number of users with different risk appetites and investment horizons is growing. Elliott and Timmermann (2016) stated that having many forecast's (valuation) users whose loss functions are heterogeneous, the provision of density forecasts (valuations) is important.

2 Literature review

Literature review on how the notion of risk may be reflected in DCF valuation may be summarised in the following classification of methods: scenarios and probabilistic cash-flows, risk-adjusted discounting, Monte Carlo based methods.
First class is based on *discrete scenarios* generated by the analyst or investor in some estimation or expert judgement process, where we may be faced with two forms: either we have some probability mass function of outcomes attached to each timestep of our time horizon or we are given some ordered outcomes labelled i.e. *best*, *base-case*, *worst* or *optimistic*, *most likely*, *pessimistic* scenarios. The latter originated from techniques used in project management and is summarized by US Navy (1958) and referred to as Programme Evaluation and Review Technique. PERT was a pragmatic solution to quickly translate qualitative labels of different scenarios into numerical representation, although the map was vague and based only on first two moments with an assumption of Gaussian data generating process. The former group of statistical methods are more idealistic, hence less common in practice, as they assume the analyst may produce discrete probability mass functions for every outcome (cash-flow) and step in the project's evaluation horizon. The literature is reach

in theoretical examples illustrating cases of dependence or independence of cash-flows over time (interested reader may confer, among others: J. C. Van Horne (1998), Damodaran Aswath (2012), Brealey, Myers, Allen, and Mohanty (2012) or seminal papers of Hillier (1963), Wagle (1967), Kazemi (1991)). However, these methods are based on the assumption of normality of the variables involved making them less useful in density forecasting, therefore some authors (i.e. Tung (1992)) suggested that some asymptotic approximations may work better to reflect kurtosis and skewness of the variables' distributions than Gaussian. The most common approximation (used in VaR calculations nowadays[1]) was proposed by Cornish, E A, Fisher (1938) and henceforth stood the test of time. The method is a bit cumbersome and does not work in all cases, but the research on the qualitative properties of Cornish-Fisher expansion (CFE) is broad and one can easily consult papers such as (Jaschke, 2002) to avoid misrepresentations. We will implement CFE both in cash-flow density and valuation density representations in the latter part of this article.

Second class may be best described as *risk-adjusment discounting*. Poterba and Summers (1995), Zenner, Berkovitz, and Clark (2009) are referring to practical side of DCF and suggest that investment managers increase the internal hurdle rate (or discount rate) beyond market-based cost-of-capital measures when evaluating projects and firms to compensate for the fact that usually the cash-flows they use are derived from so called *base-case* scenario, which tends to be too optimistic and does not allow for possibility of severe downside risk. Brealey et al. (2012) even name the practise of adjusting denominator in the discounting process as *fudge factors*. Because there is a reasonable conjecture that the forecasted cash-flows are over-estimated (or upward-biased) Ruback (2010) proposes to decrease the periodic forecasts if the omitted downside seems to be temporary or decrease the forecasts and increase the discount rate at the same time if the downside omitted is permanent. The idea falls into the same class as increasing the internal hurdle rate and gives little additional knowledge on density of the forecast or valuation, but is very common in practise. We will not use these techniques in our examples and argue that they maybe more harmful than helpful for some final users of DCF.

Finally, one may argue also that Monte Carlo method and pure stochastic cash-flows approach form a distinct class in this search for *risk* representations in DCF. The idea was popularised in financial industry by Hertz (1964) though still is less prominent than classical deterministic DCF method. In the financial literature, practical tone prevails, for example: Kelliher and Mahoney (2000) introduces different distributions to DCF scheme with a special focus on income-producing property valuation and Ali, Haddadeh, Eldabi, and Mansour (2010) use MC simulations for valuation of *internet* companies. Since Monte

[1] Modified VaR, Modified Cornish-Fisher VaR, CFVaR

Carlo simulation as such is a very deep and broadening subject it is not our ambition here to try to summarize its current state. It suffices to state that, the method, when carefully implemented may produce full density of possible valuation outcomes depending on the data generating processes' assumed form, parameters and covariance structure. Should we are able to conduct a viable stochastic discount cash-flow valuation (SDCF) we would have an informative density forecast as our end product.

The literature review was also conducted with respect to the second stage of the DCF appraisal between the author of the valuation and the final, usually heterogeneous, users. There is an obvious need to translate the density forecast into some measure that may help in investment decision process. Markowitz (1952) suggested that *the investor does consider expected return a desirable thing and variance of return an undesirable thing* , but considering only first and second moment is definitely a waste of sometimes arduous model building and simulation process. Roy (1952) in his seminal paper posits that investor will prefer safety of principal first (sometimes) requiring some minimum acceptable return and what matters the most is the R/V or reward-to-variability ratio (which concept evolved into renowned Sharpe's ratio). Since then so called *downside risk measures* flourished resulting in such measures as: below mean semivariance, below target semivariance, lower partial moments, Value-at-risk, VaR equivalent volatility.

This paper is organised as follows: Sections 3-5 treat on different sets of stochastic tools with an ultimate goal of producing a density valuation, in particular, in Section 3 we explain how one may set-up an extensive stochastic model of different variables (governed by stochastic differential equations) to be used in SDCF , Section 4 concentrates on less demanding variant of SDCF where one is given a probability mass functions of discrete possible outcomes and Section 5 is dedicated to a very practical and popular case of *three-level estimates* and their possible translations into density forecasts. In Section 6 we propose some risk measures' framework that may be implemented by individual investor or final user of (S)DCF method. Section 7 concludes and offers suggestions for further research.

3 Fully-fledged Stochastic Discounted Cash-Flows

We start from a *perfect world* situation, in which we may set-up a reasonable SDCF model from its foundations (or we are given such an extensive model), describing and designing stochastic processes of every variable (as opposed to assuming its distribution in each time-step of a considered investment horizon). For the sake of simplicity, but without a loss of our argument we assume:

1. finite horizon of forecast (valuation) $T \in \mathbb{R}_+$, hence no *terminal value*

2. D discrete time-steps of δ length[2], such that $\delta D = T$, $D \in \mathbb{N}$
3. stochastic processes governing the behaviour of selected variables as well as their covariance matrix are obtainable a prori
4. deterministic discount rate[3]
5. no taxes
6. no debt

The key step in the fully-fledged SDCF's design is to identify, which variables that influence our free cash-flows (or rather cash-flows to be discounted) are governed by stochastic processes, which are merely scaled to such variables and which are supposed to be fixed or semi-fixed. We abstract here from a certain detailed SDCF identification of variables (naming them) and we propose the following representation of cash-flows:

$$CF(t) = \sum_{i=1}^{N} R_i(t) - \sum_{i=1}^{M} C_i(t) + \sum_{i=1}^{K} S_i((R_j)_{j=1}^{N}, (C_j)_{j=1}^{M}, t) + F \tag{1}$$

where: $N, M, K \in \mathbb{N}$, R_i are the variables representing N revenue sources, C_i stands for M cost generating processes, S_i are scaled variables that depend on the whole vector of Rs and Cs with a trend, and finally F is a sum of residual cash-flows assumed constant in the model. One may argue that both revenue and cost groups of variables may fall into one class, and note that fixed cash-flows are special cases of S_i *scaled* variables. As a consequence we propose a more general form of cash-flow representation:

$$CF(t) = \sum_{i=1}^{N^*} X_i(t) + \sum_{i=1}^{K^*} S_i((X_j)_{j=1}^{N^*}, t) \tag{2}$$

where $N^* \in \mathbb{N}$ is a total number of variables for which we will define stochastic differential equations and $K^* \in \mathbb{N}$ stands for the total number of other variables and model's components which are functions of realizations of X_is and time.

In the second stage, we typically want to assume how the stochastic processes of the variables look like (intertermporal dependencies) and what are the interdependencies between variables . We start with the following general form of stochastic differential equations (SDEs) which describes the dynamics of X_i variables:

$$dX_i(t) = a_i(t, X_i)dt + b_i(t, X_i)dW_i(t) \tag{3}$$

$$dW_i dW_j = \rho_{ij} dt \tag{4}$$

[2] i.e. for 10 year horizon with quarterly time-steps we would have $D = 40$, $T = 10$, $\delta = 0.25$

[3] it may be an estimated weighted average cost of capital or any other valid rate that we require for discounting

where: $a(\cdot)$ stands for a drift function, responsible for the trend (if any), $b(\cdot)$ - a diffusion or volatility function, and $W_i(t)$ is a standard Brownian motion in $\mathcal{F}_t$ filtration. Note that we assume some covariance structure[4] of the Brownian motions driving the processes and define their matrix as Σ. Furthermore, for the sake of simplicity, we assume that the parameters that may form the drift and diffusion functions are not time dependent, hence the each random variable is homoscedastic.

Obviously, we may consider more complicated SDEs in our SDCF models by inclusion of a jump component (generated from a compound Poisson process), but we believe it would be less practical since it would be hard to find proper justification of such a choice for a certain variable in SDCF and a time series calibration could be a challenging task as well. Jump-diffusion processes are usually used in finance to add *heavier* tail-risk features, but in the case of typical variables we contemplate in discounted cash-flows method as: revenue, cost of sales, cost of materials, capital expenditures, interest payments we do not find a sound rationale to assume such a characteristics. Additionally, implementing correlation structure to a model that includes both: compound Poisson processes and standard Brownian motions may prove to be an arduous task, which we reckon, shifts the balance towards not using jump-diffusion processes in simple SDCF framework.

The proposed general SDE (4) is flexible enough to accommodate for different features needed in forecasting components of valuation, including but not limited to:

1. different forms of trend (i.e. flat, linear, time varying, quadratic) - allowing the modeller to express her views on longer term *mean* behaviour,

2. trend may be adjusted for seasonal component - especially if we use monthly or quarterly ($\delta = \{0.0833, 0.25\}$) time-steps in cash-flow forecasts,

3. geometric or arithmetical increments - as the modeller may want some processes to have log-normal distribution (i.e for non-negative realisations of sales revenues or non-positive for costs) and others - normal (i.e. sales margins, business confidence),

4. mean-reverting specifics - some processes may *hover around* some long-term average swaying away from it and coming back with weaker or stronger force,

5. a correlation structure that allows to embed assumed strengths of co-movement of certain pairs of variables (i.e. significant negative correlation between sales revenue and sales margin)

[4] which, in our particular set-up of standard Brownian motions, is in fact a correlation matrix

Since we usually want our variables to have realisations of the same sign (revenues being always positive, costs being negative etc.) our natural choices would be: geometric Brownian motion (GBM) i.e:

$$dX_i(t) = \mu_i X_i dt + \sigma_i X_i dW_i(t) \tag{5}$$

or a classic SDE that guarantees that for interest rates and additionally has a mean-reverting feature, namely, a Cox-Ingersoll-Ross (CIR) model:

$$dX_i(t) = \kappa_i \left(\theta_i - X_i(t)\right) dt + \sigma_i \sqrt{X_i(t)} dW_i(t) \tag{6}$$

where: κ_i is a speed of adjustment, θ_i - a long term mean of a process. For the processes that we want to allow to take both positive and negative values we may stick to the arithmetical Brownian motion (ABM):

$$dX_i(t) = \mu_i dt + \sigma_i dW_i(t) \tag{7}$$

or a mean-reverting Ornstein-Uhlenbeck (OU) process of a form (with the same parameters' interpretation and above):

$$dX_i(t) = \kappa_i \left(\theta_i - X_i(t)\right) dt + \sigma_i dW_i(t) \tag{8}$$

Thirdly, we need to establish some functional forms of the variables (K^* in total) that will not be directly generated by the SDEs, but will depend on their realisation at time-steps d. As there are infinitely many possible functional forms of $S_i((X_j)_{j=1}^{N^*}, t)$, we list below only a few that we posit are useful in SDCF valuation:

1. linear combinations of other variables: $S_i(X_1, X_2, ..., X_{N^*}) = \sum_{j=1}^{N^*} \alpha_j X_j$, where $\alpha_j \in \mathbb{R}$ - may be used in *percentage of sales approach* i.e. for modelling costs of sales that are proportional to the sales revenue[5].

2. piecewise constant functions: $S_i(X_j) = \sum_{r=1}^{R+1} \kappa_r \left(\mathcal{H}\left(X_j - n_{r-1}\right) - \mathcal{H}\left(X_j - n_r\right)\right)$, where κ_r are constant parameters for different intervals , $\mathcal{H}(\cdot)$ is a Heaviside function, n_r for $r = \{1, 2, ..., R, R+1\}$ is a partition of x - could be instrumental in modelling some fixed costs that depend not continuously on production or sales.

3. contingent variables: i.e. $S_i(X_1, X_2, ..., X_{N^*}) = min[0, G(X_1, X_2, ..., X_{N^*})]$, where $G(\cdot)$ may be defined as linear combination or piecewise constant functions - they maybe used to condition some additional cash-flows on stochastic variables differences, i.e. when revenues are lower than fixed costs the modeller may want to off-set any positive cash-flow generated and simulated from other variables.

[5]as described in Ali et al. (2010)

To proceed with Monte Carlo simulation in our case we would need: discretised versions of our continuous SDEs for random variables and a method to generate correlated standard Brownian motions. There are many discretisation methods used in stochastic simulation in finance (interested reader may consult Glasserman (2003), Jackel (2002) or McLeish (2005)), but for our application with relatively small number of dimensions and time-steps the most popular Euler discretisation would be the natural choice. We are interested in path-wise rather than end-point simulation of our N^* random variables because some of our dependant variables have to be recalculated in every step as they may include some assumed non-linear relationships between processes. It is easy to use Euler discretisation to our four exemplary SDEs (5, 6, 7, 8):

$$\hat{X}_{i,d\delta} = \hat{X}_{i,(d-1)\delta} + \mu_i \hat{X}_{i,(d-1)\delta}\delta + \sigma_i \hat{X}_{i,(d-1)\delta}\sqrt{\delta}\epsilon_i \qquad \text{(GBM)} \tag{9}$$

$$\hat{X}_{i,d\delta} = \hat{X}_{i,(d-1)\delta} + \kappa_i \left(\theta_i - \hat{X}_{i,(d-1)\delta}\right)\delta + \sigma_i \sqrt{\hat{X}_{i,(d-1)\delta}\delta}\epsilon_i \qquad \text{(CIR)} \tag{10}$$

$$\hat{X}_{i,d\delta} = \hat{X}_{i,(d-1)\delta} + \mu_i\delta + \sigma_i\sqrt{\delta}\epsilon_i \qquad \text{(ABM)} \tag{11}$$

$$\hat{X}_{i,d\delta} = \hat{X}_{i,(d-1)\delta} + \kappa_i \left(\theta_i - \hat{X}_{i,(d-1)\delta}\right)\delta + \sigma_i\sqrt{\delta}\epsilon_i \qquad \text{(OU)} \tag{12}$$

where $d = 1, .., D$ is a number of time-step, $\epsilon_i \sim \mathcal{N}(0, \Sigma)$ is a correlated standard normal variable via matrix Σ.

The second preparatory step to MC simulation is a choice of method to generate correlated Brownian motions. The well-established method is to use Cholesky decomposition on correlation matrix (refer to classic Glasserman (2003) for details), such that $\Sigma = C^T C$ and C is a lower triangular matrix, and then embed this C matrix in transforming simulated non-correlated standard normal variables ($z_i \sim \mathcal{N}(0, 1)$) into correlated ones (column of ϵ_i-s). In practice of SDCF modelling we may have limited number of stochastic processes $N^* \leq 3$, in which case we derive analytical formulae to change z_i-s into ϵ_iS using he assumed pair correlations ρ_{ij}:

$$\begin{cases} \epsilon_1 = z_1 \\ \epsilon_2 = \rho_{12} z_1 + \sqrt{1-\rho_{12}^2} z_2 \\ \epsilon_3 = \rho_{13} z_1 + \frac{\rho_{23} - \rho_{12}\rho_{13}}{\sqrt{1-\rho_{12}^2}} z_2 + \sqrt{\frac{1-\rho_{12}^2 - \rho_{23}^2 - \rho_{13}^2 + 2\rho_{12}\rho_{23}\rho_{13}}{1-\rho_{12}^2}} z_3 \end{cases} \tag{13}$$

The above formula works well for both $N^* = 3$ and $N^* = 2$, where in the latter case one should ignore the third equation.

Last but not least, the modeller may decide on imposing additional restrictions and adjustments on a simulated path cash-flows. For example, one may want to zero-out all consecutive cash-flows in a path after a certain cash-flow generated shows nonpositive values, to reflect the default scenarios as well.

The result of Monte Carlo simulation in a set-up generally described above, would be B_{sim} generated paths of combined $CF(t)$ variables, each path containing a sample of possible cash-flows evolution. The series of CF values in each path should be discounted (recall that we assumed deterministic discounting rates availability for the modeller) and the collection of B_{sim} sums of discounted cash-flows is a basis for the desired density forecast we will use in Chapter 5 for risk assessment purposes.

The fully-fledged N^*-factor SDCF model (henceforth: FFSDCF) overcomes the main disadvantages of classic DCF with sensitivities as described by Ali et al. (2010) or Kelliher and Mahoney (2000) because it takes into account the interdependencies between variables and captures the whole probability densities of possible realisations of different elementary variables as well as of the final valuation.

4 Towards simplified SDCF

In this section we use the nomenclature and definitions set in the previous part of this paper. Since proper specification of stochastic differential equations may be arduous and cumbersome in some cases, the modeller may want to express hers views on variables X_i behaviour in the future by deciding whether to take discrete or continuous approach to them and deciding which specification to follow:

1. defining distributions (and their parameters) of variables X_i in every time-step d of the forecast (D distributions for each of N^* variables). In that case the trend is implicit in the time evolution of the distributions.

2. defining distributions (and their parameters) of variables' increments ΔX_i, the same for every time-step d of the forecast (one distribution for each of N^* variables). In such case additional explicit assumption on trend component is needed, as well as the size of increment Δ_i for each variable.

Unquestionable advantage of the first approach is the flexibility of implementing any idea about future behaviour of different variables in every time-step via different distributions (i.e. normal, lognormal, Student's t, uniform, triangle, beta, gamma, Pearson etc). The application scope however seems to be limited due to the rising complexity even in *normal* DCF horizons of, say, 10 time-steps with 3 stochastic variables, let alone more extensive set-ups of SDCF. One way to make this method more implementable is to build the model on discrete distributions of evolved variables using *standardised* probability mass functions with limited number (henceforth: Q) of points[6]. Our task is then to define for each i and

[6] usually some small, odd number is considered: 3, 5, 7

each d the mass points $x_{i,d\delta,q}$ and their corresponding standardised probability p_q.

$$P(X_{i,d\delta} = \{x_{i,d\delta,1}, ..., x_{i,d\delta,q}, ..., x_{i,d\delta,Q}\}) = \{p_1, ..., p_q, ..., p_Q\} \tag{14}$$

where $\Sigma_{q=1}^{Q} p_q = 1$ and we read it $P(X_{i,d\delta} = x_{i,d\delta,q}) = p_q$ for every $q \in Q$. In such case, the modeller retains some flexibility of shaping probability densities[7], but obviously loosing the whole catalogue of named distributions' characteristics and hence sometimes economic interpretations. Another drawback is the discreteness of the *standardised* distributions, which at the end result in a *sparse histogram* of possible valuations rather than the *density* valuation we wish to obtain. Hence some transformation (i.e. calculating moments and performing Cornish-Fisher expansions or using kernel density estimation) of the final simulation object will be necessary before using it in the individual risk assessment.

The second approach is closer in its origins to the fully-fledged SDCF shown in the previous, dedicated section. The main simplification is that we have to design one distribution which governs the diffusion in each time-step and one trend function per variable instead of $N^* \times D$ distributions. For the discrete distributions we would have to define for each variable X_i:

$$P(\Delta X_i = \{\Delta x_{i,1}, ..., \Delta x_{i,q}, ..., \Delta x_{i,Q}\}) = \{p_1, ..., p_q, ..., p_Q\} \tag{15}$$

the same for every time-step $d \in D$. This simplification leads to much faster program-to-compute times than the first method, but one should bear in mind that the discretised trend should be explicitly additionally assumed here[8]. In case of continuous distributions we would like to set for each variable:

$$\Delta X_i \sim \mathcal{D}(\Theta) \tag{16}$$

where $\mathcal{D}(\cdot)$ is an assumed distribution and Θ is a vector of particular parameters of such an object. In these decisions the method is comparable with the fully-fledged SDCF from Section 3, but one should carefully choose the distributions' forms to be conveniently invertible and for which fast algorithms of inversion exist.

Correlation structure of our variables or their increments in environments of diversified distributions (not necessarily all normal) may cause the simulation to be more demanding and complex task than in the case of SDCF from Section 2. Obviously, only if all the variables are orthogonal to each other then we have a simpler calculation. Unfortunately, in other cases of several variables following different distributions we have to use copulas of inverted distributions of these variables (therefore uniform on $[0, 1]$) or simulate correlated

[7] through both: vector of probabilities and distances between certain mass points $x_{i,d\delta,q}$

[8] discretised drift examples were shown in Section 2

standard normal variables and then convert every variable to the desired final distribution using proper probability integral transform (PIT)[9]. The latter algorithm is sometimes called *NORmal To Anything* (NORTA), and as Xiao (2017) indicated the main challenge when discrete distributions are also involved is to determine suitable correlation coefficient in normal space for a specified (by a modeller) correlation coefficient. Deeper review of the techniques necessary to simulate heterogeneous (by distribution) correlated variables is out of scope of this article and we refer interested reader to the works of Chen (2001), Lebrun and Dutfoy (2009) or Madsen and Birkes (2013).
Bearing our goal of presenting computationally simpler model in mind, we posit that *standardised* probability mass functions with some 5-7 weight points defined for every orthogonal stochastic variable and every time-step may lead to simpler that full SDCF implementations (henceforth: SSDCF) . The literature review showed that both discretisation of some variables' distributions as well as allowing them to be from different classes, comes with a price of higher complexity of simulation when the variables are supposed to be not orthogonal, therefore in search for less involving calculation we should contemplate the assumption of our SDCF model variables' orthogonality.

5 Three-level estimates SDCF

In search for even simpler representation of stochastic cash-flows in DCF method we may turn to the, so called, three-level estimates[10] described in Section 2. This method may be treated as a subclass of the specification (SSDCF) we detailed in the previous section with the following differences:

1. we have three mass points for each stochastic variable
2. the probability mass function has to be specified with respect to probabilities, as we have usually only qualitative labels for different scenarios: *best, base, worse* etc.

The mathematical representation of the three-level SDCF (henceforth: 3LSDCF) model's mass probability function is:

$$P(X_{i,d\delta} = \{x_{i,d\delta,\text{best}}, x_{i,d\delta,\text{base}}, x_{i,d\delta,\text{worse}}\}) = \{p_{\text{best}}, p_{\text{base}}, p_{\text{worst}}\} \tag{17}$$

which shall be defined for every variable X_i and every time-step $d \in D$. The vector of probabilities $\mathbf{p} = \{p_{\text{best}}, p_{\text{base}}, p_{\text{worst}}\}$ should be specified as well. The literature and practice point to different possible representations, i.e $\mathbf{p} = \{\frac{1}{6}, \frac{2}{3}, \frac{1}{6}\}$ in PERT, $\mathbf{p} = \{\frac{1}{4}, \frac{1}{2}, \frac{1}{4}\}$ in J. Van

[9] which is actually a usage of a particular copula - namely Gaussian copula
[10] one of the first usage of this name is found in Hertz (1964)

Horne (1966), whereas the final decision is left to the particular problem's modeller. There are some practical aspects of this method to consider:

1. for the sake of simplicity we need to assume orthogonality of the variables[11]

2. since we have to establish $x_{i,d\delta,\text{best}}, x_{i,d\delta,\text{base}}, x_{i,d\delta,\text{worse}}$ of the longer and longer forecast horizons (d) we have to reflect the implied variance of the data generated process somehow [12]

3. the resulting from Monte Carlo simulation set of valuations will take very limited number of values due to extensive discretisation of every variable involved, hence there will be a need for some additional transformation (before mentioned: Cornish-Fisher expansions or kernel densities) of the final results to acquire density forecast.

6 Assessing risk from individual user's perspective using density valuations

Firstly, we discuss the resultant density object and possible transformations for every model we discussed. As we have already signalled, only in the case of the fully-fledged SDCF the result of Monte Carlo simulation will give the modeller the access to *almost* continuous distribution of valuation outcomes. In two other specifications the product of simulation is rather sparse histogram, which usually is not suitable for density forecast considerations. However there are at least two groups of methods of smoothing out the valuation set, namely: kernel density estimation and Cornish-Fisher approximation (Fisher and Cornish (1960)). For the sake of completeness, we briefly explain how these tools may be instrumental in our goal.

Kernel density estimation (KDE) is a widely used smoothing method in forecasting (reader is may consult a seminal paper of Hyndman, Bashtannyk, and Grunwald (1996) and some later works of Harvey and Oryshchenko (2012)). The cornerstone of KDE is a local density estimator $\hat{f}_h(v)$ with a given smoothing parameter $h > 0$ (sometimes called: bandwidth):

$$\hat{f}_h(v) = \frac{1}{Bh} \sum_{i=1}^{B} K\left(\frac{v - v_i}{h}\right) \tag{18}$$

where B is a number of elements in a set V_{SDCF}, h is a bandwidth used to build the kernel (it may be estimated from data or we may use some rule-of-thumb to infer it), $K(\cdot)$ is a kernel function (non-negative and such that $\int_{\mathbb{R}} K(x)dx = 1$) such as: normal, Epanechnikov,

[11] see comments in the previous section

[12] it would be natural to expect that the distance $|x_{i,d\delta,\text{best}} - x_{i,d\delta,\text{worse}}|$ grows as d increases.

bi-weight, tri-weight, triangular etc. We reckon *normal kernels* are appropriate to our application because of their unrestricted domain, the characteristics which is particularly useful in filling gaps of incomplete information:

$$K(x) = \frac{1}{\sqrt{2\pi}} e^{\frac{1}{2}x^2} \tag{19}$$

The choice of smoothing parameter h has a strong influence on the shape of our density function. The higher h the smoother the density would turn to be, in the limit morphing to normal density. The lower the parameter the more ragged the density would be, hence some balance in the modeller's decision is needed. We claim that for our application to products of SDCF simulations - the so-called *Silverman's* rule is enough to follow[13].
Cornish-Fisher approximation (CFE) represents a different approach[14] to smoothing based on calculation of four first moments of a given sample. These non-central moments are used in CPE to estimate quantile function of the population. As (Tung, 1992), Jaschke (2002) suggest the method is cumbersome because monotonicity of the cumulative distribution as well as convergence is not guaranteed. CPE is more accurate when returns ale close to normal, which may not be the case in our valuations. It is worth underlying that we would rather not expect Gaussian distribution of valuations, especially when there are more than one stochastic processes and we have some state-contingent elements of the cash flow to be discounted. Jaschke (2002) showed some *safe* intervals of adjusted third and forth moments for which the method works sufficiently good. Since the method requires more attention and care than KDE, we believe that the latter is more suitable to our end.
Secondly, having density distribution of valuations from FFSDCF or the transformed results of SSDCF and 3LSDCF in the manner described above, one may go far beyond obvious and simplifying measures (expected value and variance) and calculate precise probabilities of different outcomes and compare them with some objectives or a *line of defence*. To this end, it is easy to construct a quantile function to assess the probability of a certain valuation to fall below a prescribed level:

$$Q(p, V_{SDCF}) = \inf\{v \in V_{SDCF}; p \leq F_{SDCF}(v)\} \tag{20}$$

where V_{SDCF} is a set of all generated valuations v in a certain model and $F_{SDCF}(\cdot)$ is the cumulative distribution function. Roy (1952) posited that *an investor will prefer safety of principal first* and that she would set some minimum acceptable return on an investment

[13]Silverman (1986) claims that the optimal bandwidth that minimises the mean integrated squared error is given by $h_{opt} = \left(\frac{4\hat{\sigma}}{3B}\right)^{\frac{1}{5}} \approx 1.06\hat{\sigma}B^{-\frac{1}{5}}$ and $\hat{\sigma}$ is a standard deviation of a sample

[14]some other methods to estimate a quantile function of a distrubution from a given sample: Solomon-Stephens approximation, Johnson transformation, saddle-point approxination, Fourrier inversion (after: Jaschke (2002))

made, called *disaster level.* As Nawrocki (2000) suggests, Roy claims that the investor would prefer the investment with the smallest probability of going below this *disaster* level. Having the density valuation in hand we may modify and extend this framework to assess risk from the perspective of an individual user in the following way:

1. define a *return transformation* of a valuation set by:

$$t(v, v_0) = \left\{ v \in V_{SDCF} \rightarrow r \in \mathbb{R}; \quad r = \frac{v}{v_0} - 1; \quad v_0 \in \mathbb{R} \quad \wedge \quad v_0 \neq 0 \right\} \tag{21}$$

   where v_0 is a reference point valuation, i.e current market price $v_{mkt,d=0}$ of the investment[15].

2. let $\tilde{V}_{SDCF}$ be the transformed set of returns and $\tilde{F}_{SDCF}(\cdot)$ the cumulative distribution function of transformed, in such a way, variable.

3. lets call *stop-loss* (t_{sl}) the reference negative return $t(v, v_{mkt,d=0})$ at which the potential investor would have to terminate the investment with a loss in avoidance of making even bigger losses on this investment in her subjective judgement[16]

4. lets call *risk-free target* (t_{rf}) the reference return $t(v, v_{mkt,d=0})$ which equals to the return on investment of $v_{mkt,d=0}$ units in risk-free instruments made for the a limited time horizon $h \leq T$, during which the investor assumes the valuation will materialize.

5. lets call *break-even* (t_{be}) the reference return which equals to zero[17].

Then we may calculate the probability of exceeding the *stop-loss* return to the downside as equal to $\tilde{F}_{SDCF}(t_{sl})$, probability of making losses on the investment $\tilde{F}_{SDCF}(be)$ and the probability of not making at least risk-free rate $\tilde{F}_{SDCF}(t_{rf})$, which measures clearly contribute to solving the incomplete information problem in case of some three-level PERT-alike scenarios with sensitivity analysis. Probability of missing the expected value to the downside is not very interesting as it is by the definition 0.5, but we propose here some measures that may help the user (investor) in evaluating the character of a certain model on the top of classic four moments (mean, variance, skewness and kurtosis):

1. *implicit optimism ratio*, which may be defined as probability of non negative valuation to the probability of not hitting the stop-loss level[18]:

$$\Omega = \frac{1 - \tilde{F}_{SDCF}(be)}{1 - \tilde{F}_{SDCF}(t_{sl})} \tag{22}$$

[15] it may be an asking price in non-public auction as well, not necessarily the exchange traded instrument's price

[16] we believe that is closer to the Roy's idea of *disaster level*

[17] it may differ from t_0 by transactional costs influence

[18] for some non-tradable investments *stop-loss* level may not be feasible, hence we will take $\tilde{F}_{SDCF}(t_{sl}) = 0$

2. *market efficiency distance*[19], which evaluates difference in probability of achieving the expected valuation (0.5) and the one of achieving better that risk-free results (if the markets are effective and complete, but the first asset pricing theorem this difference should be close to zero)

$$\mathcal{E} = |0.5 - \tilde{F}_{SDCF}(t_{rf})| \tag{23}$$

3. *reservation valuation* at different percentiles: i.e. $Q(0.01, \tilde{V}_{SDCF})$, $Q(0.05, \tilde{V}_{SDCF})$ or $Q(0.10, \tilde{V}_{SDCF})$.

We may go even a step forward and use utility theory to judge if some investment proposal we consider in DCF is appropriate for a risk profile of a particular investor. Kahneman and Tversky (1979) in his prospect theory presented and idea that may be particularly interesting in our analysis. It may be rephrased in the *overall utility* measure:

$$\mathcal{U} = \int_{\mathbb{R}} U(\tilde{v})\hat{f}(\tilde{v})d\tilde{v} \tag{24}$$

where $\hat{f}(\tilde{v})$ is a density function obtained in the simulations and transformations described before, and a special (in our case continuous) utility function is of a form:

$$U(\tilde{v}) = \tilde{v}^{\alpha}\mathbb{1}_{\{\tilde{v}\geq 0\}} - \lambda(-\tilde{v})^{\beta}\mathbb{1}_{\{\tilde{v}<0\}} \tag{25}$$

where $\alpha, \beta > 0$ are shape parameters and λ is a loss aversion parameter (in original works of Kahneman and Tversky (1979) we may found $\alpha = \beta = 0.88$ and $\lambda = 2.25$)
Therefore the density valuation as a result of SDCF Monte Carlo simulation may lay a good basis for variety of risk measures that may be individualised to particular user's characteristics. One valuation set may yield in different decisions for distinctive investors.

7 Conclusions and further research

We discussed different possibilities of constructing density valuations and their application in individualised investment risk assessment from the view point of a final user of a certain stochastic discounted cash-flow model. The review of stochastic techniques from a practical perspective of DCF modeller showed that there are some choices and decisions to be made during a model design phase. The space of degrees of freedom includes among others: framework of a model[20], variables assumed to be stochastic, forms of SDEs for each stochastic variable, functional forms for non-stochastic variables, parameters of SDEs and

[19] or: model's incompleteness
[20] FFSDCF, SSDCF, 3LSDCF

other functions used to define variables, correlation structure or smoothing methods for the final density valuation.

In every framework contemplated in this article it is feasible to generate density forecast object, which may be used in investment risk assessment process made by individuals and be the basis for a quantile function, higher moments calculation, stop-loss level set-up an evaluation, proposed measures of: implicit optimism ration, market efficiency distance, reservation valuations or even an overall utility in line with prospect theory.

Further research may be also conducted on calibration of a certain SDCF framework to default risk parameters observed in the market of corresponding financial instruments. One may argue that density valuation of a share should incorporate a default risk of that company observed in its issued debt. We believe that there are ways to establish such a relationship in stochastic frameworks of SDCF by manipulating mass of negative outcomes or shifting the whole cumulative density function to the left. Moreover, from the regulatory perspective on the relation between retail clients and financial institutions, in particular, the suitability of different products to their needs, risk profile and knowledge, it would be wise to standardise a group of risk measures and aggregate utility functions to be used as a simplified gauges of appropriateness of a certain investment proposal to a particular investor with a particular risk profile.

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